QNo→ To State and Prove the integral test.

Ans.→ Statement:- If $f(x)<0$, when $x>0$ and if $f(x)$ decreases as x increases, then the sequences. when $x>0$ and if $f(x)$ decreases as x increases, then the sequences

$$S_n = f(1)+f(2)+\text{------}+f(n),\ n=1,2,3,\text{-----}$$

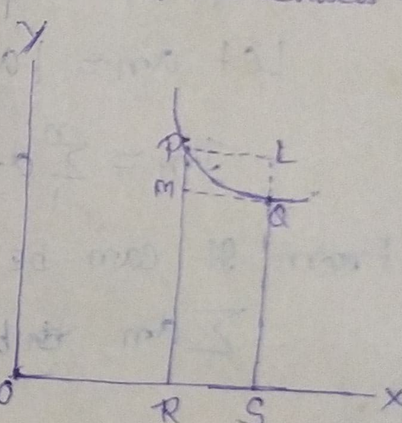
$$\& \quad I_n = \int_1^n f(x)\,dx,\ (n=1,2,3\text{-----}).$$

are either both Convergent or both divergent.

Proof:- Let $S_n = f(1)+f(2)+\text{------}+f(n)$

$$I_n = \int_1^n f(x)\,dx.$$

Since $f(x)>0$, when $x>0$, the Sum S_n decreases and I_n increases as n increases.



Let P, Q be two Points on the curve

$$y=f(x) \text{ --- } (1)$$

Such that $OR=n$ and $OS=n+1$ where R, S are the feet of the ordinates at P & Q. Complete the rectangle PMQL. Then rect. MRSQ < area PQRS < rect. PRSL. Since $RS=1$, $PR=f(n)$ & $QS=f(n+1)$ we get from (1)

$$f(n+1) < \int_n^{n+1} f(x)\,dx < f(n) \text{ --- } (2)$$

writing $n=1,2,3,\text{-----},n-1$ in (1) and adding, we get

$$S_n - f(1) < I_n < S_n - f(n) \text{ --- } (3)$$

Now, Suppose that $I_n \to$ a finite limit l as $n\to\infty$. Since $\{I_n\}$ is monotonic increasing sequence, we have $I_n \leq l$ and so from (3)

$$S_n < I_n + f(1) \leq l + f(1) \quad \text{--- (4)}$$

But $l + f(1)$ is independent of n, so that by (4), $\{S_n\}$ is a monotonic increasing sequence whose upper bound $\leq l + f(1)$. Hence $S_n \to$ a finite limit, say S such that $S = l + f(1)$

Similarly, if $S_n \to S$, (3) gives $I_n < S_n - f(n) < S_n \leq S$

$$I_n \to l \leq S.$$

If $I_n \to \infty$, then since $S_n > I_n + f(n)$, $S_n \to \infty$, and if $S_n \to \infty$, then since $I_n > S_n - f(1)$, $I_n \to \infty$.

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